Exercise 1

The components of a symmetric tensor $\boldsymbol{\tau}$ are

$$\begin{array}{lll} \tau_{xx} = 3 & \tau_{xy} = 2 & \tau_{xz} = -1 \\ \tau_{yx} = 2 & \tau_{yy} = 2 & \tau_{yz} = 1 \\ \tau_{zx} = -1 & \tau_{zy} = 1 & \tau_{zz} = 4 \end{array}$$

The components of a vector $\mathbf{v}$ are

$$v_x = 5 \quad v_y = 3 \quad v_z = -2$$

Evaluate

(a) $[\boldsymbol{\tau} \cdot \mathbf{v}]$ **(b)** $[\mathbf{v} \cdot \boldsymbol{\tau}]$ **(c)** $(\boldsymbol{\tau} : \boldsymbol{\tau})$
(d) $(\mathbf{v} \cdot [\boldsymbol{\tau} \cdot \mathbf{v}])$ **(e)** $\mathbf{vv}$ **(f)** $[\boldsymbol{\tau} \cdot \boldsymbol{\delta}_x]$

Solution

(a) $$\boldsymbol{\tau} \cdot \mathbf{v} = \left(\sum_{i=1}^{3}\sum_{j=1}^{3} \boldsymbol{\delta}_i \boldsymbol{\delta}_j \tau_{ij}\right) \cdot \left(\sum_{k=1}^{3} \boldsymbol{\delta}_k v_k\right) = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \boldsymbol{\delta}_i (\boldsymbol{\delta}_j \cdot \boldsymbol{\delta}_k) \tau_{ij} v_k = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \boldsymbol{\delta}_i \delta_{jk} \tau_{ij} v_k$$

$$= \sum_{i=1}^{3}\sum_{j=1}^{3} \boldsymbol{\delta}_i \tau_{ij} v_j = \sum_{i=1}^{3} \boldsymbol{\delta}_i (\tau_{i1} v_1 + \tau_{i2} v_2 + \tau_{i3} v_3)$$

$$= \boldsymbol{\delta}_1 (\tau_{11} v_1 + \tau_{12} v_2 + \tau_{13} v_3)$$

$$+ \boldsymbol{\delta}_2 (\tau_{21} v_1 + \tau_{22} v_2 + \tau_{23} v_3)$$

$$+ \boldsymbol{\delta}_3 (\tau_{31} v_1 + \tau_{32} v_2 + \tau_{33} v_3)$$

$$= \boldsymbol{\delta}_1 (15 + 6 + 2) + \boldsymbol{\delta}_2 (10 + 6 - 2) + \boldsymbol{\delta}_3 (-5 + 3 - 8)$$

$$= 23\boldsymbol{\delta}_1 + 14\boldsymbol{\delta}_2 - 10\boldsymbol{\delta}_3$$

$$= 23\boldsymbol{\delta}_x + 14\boldsymbol{\delta}_y - 10\boldsymbol{\delta}_z$$

(b)

$$\mathbf{v}\cdot\boldsymbol{\tau} = \left(\sum_{i=1}^{3}\boldsymbol{\delta}_i v_i\right)\cdot\left(\sum_{j=1}^{3}\sum_{k=1}^{3}\boldsymbol{\delta}_j\boldsymbol{\delta}_k\tau_{jk}\right) = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}(\boldsymbol{\delta}_i\cdot\boldsymbol{\delta}_j)\boldsymbol{\delta}_k v_i\tau_{jk} = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\delta_{ij}\boldsymbol{\delta}_k v_i\tau_{jk}$$

$$= \sum_{j=1}^{3}\sum_{k=1}^{3}\boldsymbol{\delta}_k v_j\tau_{jk} = \sum_{j=1}^{3}(\boldsymbol{\delta}_1 v_j\tau_{j1} + \boldsymbol{\delta}_2 v_j\tau_{j2} + \boldsymbol{\delta}_3 v_j\tau_{j3})$$

$$= \boldsymbol{\delta}_1\sum_{j=1}^{3}v_j\tau_{j1} + \boldsymbol{\delta}_2\sum_{j=1}^{3}v_j\tau_{j2} + \boldsymbol{\delta}_3\sum_{j=1}^{3}v_j\tau_{j3}$$

$$= \boldsymbol{\delta}_1(v_1\tau_{11} + v_2\tau_{21} + v_3\tau_{31})$$

$$+ \boldsymbol{\delta}_2(v_1\tau_{12} + v_2\tau_{22} + v_3\tau_{32})$$

$$+ \boldsymbol{\delta}_3(v_1\tau_{13} + v_2\tau_{23} + v_3\tau_{33})$$

$$= \boldsymbol{\delta}_1(15 + 6 + 2) + \boldsymbol{\delta}_2(10 + 6 - 2) + \boldsymbol{\delta}_3(-5 + 3 - 8)$$

$$= 23\boldsymbol{\delta}_1 + 14\boldsymbol{\delta}_2 - 10\boldsymbol{\delta}_3$$

$$= 23\boldsymbol{\delta}_x + 14\boldsymbol{\delta}_y - 10\boldsymbol{\delta}_z$$

(c)

$$\boldsymbol{\tau} : \boldsymbol{\tau} = \left(\sum_{i=1}^{3}\sum_{j=1}^{3} \boldsymbol{\delta}_i\boldsymbol{\delta}_j\tau_{ij}\right) : \left(\sum_{k=1}^{3}\sum_{l=1}^{3} \boldsymbol{\delta}_k\boldsymbol{\delta}_l\tau_{kl}\right) = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}(\boldsymbol{\delta}_i\boldsymbol{\delta}_j : \boldsymbol{\delta}_k\boldsymbol{\delta}_l)\tau_{ij}\tau_{kl}$$

$$= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}(\boldsymbol{\delta}_i \cdot \boldsymbol{\delta}_l)(\boldsymbol{\delta}_j \cdot \boldsymbol{\delta}_k)\tau_{ij}\tau_{kl}$$

$$= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\delta_{il}\delta_{jk}\tau_{ij}\tau_{kl}$$

$$= \sum_{i=1}^{3}\sum_{j=1}^{3}\tau_{ij}\tau_{ji}$$

$$= \sum_{i=1}^{3}(\tau_{i1}\tau_{1i} + \tau_{i2}\tau_{2i} + \tau_{i3}\tau_{3i})$$

$$= \sum_{i=1}^{3}\tau_{i1}\tau_{1i} + \sum_{i=1}^{3}\tau_{i2}\tau_{2i} + \sum_{i=1}^{3}\tau_{i3}\tau_{3i}$$

$$= \tau_{11}\tau_{11} + \tau_{21}\tau_{12} + \tau_{31}\tau_{13}$$

$$+ \tau_{12}\tau_{21} + \tau_{22}\tau_{22} + \tau_{32}\tau_{23}$$

$$+ \tau_{13}\tau_{31} + \tau_{23}\tau_{32} + \tau_{33}\tau_{33}$$

$$= 9 + 4 + 1 + 4 + 4 + 1 + 1 + 1 + 16$$

$$= 41$$

(d) $\mathbf{v}\cdot[\boldsymbol{\tau}\cdot\mathbf{v}] = (5\boldsymbol{\delta}_1 + 3\boldsymbol{\delta}_2 - 2\boldsymbol{\delta}_3)\cdot(23\boldsymbol{\delta}_1 + 14\boldsymbol{\delta}_2 - 10\boldsymbol{\delta}_3) = 5\cdot 23 + 3\cdot 14 + (-2)(-10) = 177$

(e)
$$\begin{aligned}\mathbf{vv} = \left(\sum_{i=1}^{3}\boldsymbol{\delta}_i v_i\right)\left(\sum_{j=1}^{3}\boldsymbol{\delta}_j v_j\right) &= (\boldsymbol{\delta}_1 v_1 + \boldsymbol{\delta}_2 v_2 + \boldsymbol{\delta}_3 v_3)(\boldsymbol{\delta}_1 v_1 + \boldsymbol{\delta}_2 v_2 + \boldsymbol{\delta}_3 v_3)\\
&= v_1^2\boldsymbol{\delta}_1\boldsymbol{\delta}_1 + v_1v_2\boldsymbol{\delta}_1\boldsymbol{\delta}_2 + v_1v_3\boldsymbol{\delta}_1\boldsymbol{\delta}_3\\
&\quad + v_2v_1\boldsymbol{\delta}_2\boldsymbol{\delta}_1 + v_2^2\boldsymbol{\delta}_2\boldsymbol{\delta}_2 + v_2v_3\boldsymbol{\delta}_2\boldsymbol{\delta}_3\\
&\quad + v_3v_1\boldsymbol{\delta}_3\boldsymbol{\delta}_1 + v_3v_2\boldsymbol{\delta}_3\boldsymbol{\delta}_2 + v_3^2\boldsymbol{\delta}_3\boldsymbol{\delta}_3\\
&= 25\boldsymbol{\delta}_x\boldsymbol{\delta}_x + 15\boldsymbol{\delta}_x\boldsymbol{\delta}_y - 10\boldsymbol{\delta}_x\boldsymbol{\delta}_z\\
&\quad + 15\boldsymbol{\delta}_y\boldsymbol{\delta}_x + 9\boldsymbol{\delta}_y\boldsymbol{\delta}_y - 6\boldsymbol{\delta}_y\boldsymbol{\delta}_z\\
&\quad - 10\boldsymbol{\delta}_z\boldsymbol{\delta}_x - 6\boldsymbol{\delta}_z\boldsymbol{\delta}_y + 4\boldsymbol{\delta}_z\boldsymbol{\delta}_z\end{aligned}$$

(f)
$$\begin{aligned}\boldsymbol{\tau}\cdot\boldsymbol{\delta}_x = \left(\sum_{i=1}^{3}\sum_{j=1}^{3}\boldsymbol{\delta}_i\boldsymbol{\delta}_j\tau_{ij}\right)\cdot\boldsymbol{\delta}_1 = \sum_{i=1}^{3}\sum_{j=1}^{3}\boldsymbol{\delta}_i(\boldsymbol{\delta}_j\cdot\boldsymbol{\delta}_1)\tau_{ij} = \sum_{i=1}^{3}\sum_{j=1}^{3}\boldsymbol{\delta}_i\delta_{j1}\tau_{ij} &= \sum_{i=1}^{3}\boldsymbol{\delta}_i\tau_{i1}\\
&= \boldsymbol{\delta}_1\tau_{11} + \boldsymbol{\delta}_2\tau_{21} + \boldsymbol{\delta}_3\tau_{31}\\
&= 3\boldsymbol{\delta}_1 + 2\boldsymbol{\delta}_2 - \boldsymbol{\delta}_3\\
&= 3\boldsymbol{\delta}_x + 2\boldsymbol{\delta}_y - \boldsymbol{\delta}_z\end{aligned}$$